Perturbed Friedmann Cosmologies Filled with Dust and Radiation

Z. Perjés

KFKI Research Institute for Particle and Nuclear Physics,
H-1525 Budapest 114, P.O.Box 49, Hungary

February 27, 2001

ABSTRACT

An infinite number of perturbed $k = 0$ Friedmann cosmologies filled with dust and radiation is found. As we go up the sequence, the solutions contain an increasing number of integration functions. With the coordinate gauge adopted to co-move with the perturbed matter, the solution of a pair of coupled equations for the trace h of the metric perturbations and for the radiation density perturbations $\delta\rho_r$ is the key to the problem. An estimate is given of the temperature variation of the cosmic microwave background radiation due to the Sachs-Wolfe effect. It is found that the temperature fluctuations can grow faster than in the absence of radiation.

1. Introduction

The first-order perturbations of a Friedmann universe with a flat 3-space filled either with dust or radiation have been obtained by Sachs and Wolfe 1967. They computed the temperature fluctuations of the cosmic background radiation in the dust-filled universe, assuming that the domain in which the photon travels is matter-dominated. This classic prediction for the temperature fluctuations overestimates the experimental value (Mather 1992) by several orders of magnitude. The prediction has been made by considering only the gravitational perturbations along the path of the photon. Obviously, the inclusion of random temperature fluctuations on the surface of the last scattering can only increase the effect. Neither did the discovery of large-scale structures (voids and walls) bring us closer to the resolution of this paradox. Though it has been put forward (Mather 1992) that the observed fluctuations are primordial (as opposed to propagation effects), this is hard to take seriously unless we are able to reduce the magnitude of the Sachs-Wolfe effect.

Here we compute the perturbations of the $k = 0$ Friedmann universe in the presence of *both* dust and radiation. Perturbations of two-fluid cosmologies have been studied in the past decades (Kodama 1987, Koranda 1994, Perjés and Komárik 1995). Instabilities are known to exist in two-fluid cosmologies (Mukhanov 1992). One could reduce the magnitude of the metric fluctuations affecting the photon orbits by assuming, for example, that the observed large structures are late emerging manifestations of the instabilities. The main result of the present paper, in Sec. 6, is an infinite series of solutions, with an increasing number of integration functions. Like in the pure dust model, the solutions contain a relatively increasing and a relatively decreasing mode of the radiation density perturbation. The growth is, however faster in the presence of radiation and even faster as we go up the sequence of our solutions. In Sec. 7, we estimate the magnitude of the Sachs-Wolfe effect on the temperature variation of the cosmic background radiation. We find that the relatively growing mode is incompatible with the observations already with the first solution.

The energy-momentum tensor is a sum of those of the two media,

$$T^a_b = T^{\ a}_{m\,b} + T^{\ a}_{r\,b} \tag{1}$$

The contribution of the dust has the form

$$T^{\ a}_{m\,b} = \rho_m u^a u_b \ . \tag{2}$$

For the radiation,

$$T^{\ a}_{r\,b} = \tfrac{4}{3}\rho_r u^a u_b - \tfrac{1}{3}\rho_r \delta^a_{\ b}. \tag{3}$$

The four-velocities are normalized $u^a u_a = 1$. We assume that, after the decoupling, the conservation law $T^a_{b;a} \equiv (\rho + p)\, u_{a;b} u^b - p_{,a} + u_a u^b p_{,b} = 0$ applies separately to both the matter and the radiation components:

$$T^a_{mb;a} = 0, \qquad T^a_{rb;a} = 0. \tag{4}$$

This is justifiable because the decoupling occurs near the time of equal matter-and-radiation density, and after that, coupling occurs only via the universal gravitational interaction. We get two energy conservation equations by transvecting with u^b,

$$(\rho_m u^a_m)_{;a} = 0, \qquad \left(\rho_r^{3/4} u^a_r\right)_{;a} = 0. \tag{5}$$

and two momentum conservation laws:

$$u_{m[a,b]} u^b_m = 0, \qquad 4\rho_r \left(u_{ra,b} - u_{rb,a}\right) u^b = \rho_{r,a} - u_{ra} u^b_r \rho_{r,b}. \tag{6}$$

2. The perturbed model

We adopt the conformal form of the metric

$$g_{ab} = a^2(\eta)\,(\eta_{ab} + h_{ab})\,, \tag{7}$$

where $(\eta_{ab}) = diag\,(1,-1,-1,-1)$ and h_{ab} is the metric perturbation. The indices of the perturbed quantities are lowered and raised by the Minkowski metric $\eta_{ab} = \eta^{ab}$. We shall not use the explicit form of the scaling function $a(\eta)$ as long as possible. In the perturbed universe, $h_{ab} \neq 0$, the densities of the components can be written to first order

$$\rho_i^{(1)} = \rho_i + \delta\rho_i\ . \tag{8}$$

where i stands either for m (matter) or r (radiation). Here ρ_r and ρ_m are the unperturbed densities

$$\rho_m(\eta) = \rho_{m0}\frac{a_0^3}{a^3}\,, \qquad\qquad \rho_r(\eta) = \rho_{r0}\frac{a_0^4}{a^4} \tag{9}$$

and $\delta\rho_i$ are the first-order density perturbations. According to the unperturbed models, the radiation density ρ_r dies out faster than the matter density, ρ_m .

The dipole effect of the cosmic background radiation provides an experimental value for the local relative velocity of the order of 600 km/sec. Thus we have good reason to assume that $\delta u_m^i \neq \delta u_r^i$. Here we stick to a comoving gauge. For gauge invariant methods, *cf.* Refs. Magueijo 1993, Russ 1993, Kodama 1987, Dunsby 1991, Mukhanov 1992. We choose coordinates comoving with the matter:

$$u_m^a = u_0^a \tag{10}$$
$$u_r^a = u_0^a + \delta u_i^a \tag{11}$$

where the coincident unperturbed velocities are

$$u_0^a = \frac{1}{a}\delta_0^a\ . \tag{12}$$

The normalization conditions imply that $h_{00} = 0$ and $\delta u_r^0 = \ \delta u_{r0}\ = 0$.

The permissible coordinate transformations in the comoving gauge are (Sachs and Wolfe 1967)

$$\widehat{x}^a = x^a - \xi^a$$

where the first-order function ξ^a has the form

$$\xi^0 = \frac{b(x^\beta)}{a}, \qquad \xi^\alpha = c^\alpha(x^\beta) \tag{13}$$

and x^α are the space coordinates for $\alpha = 1, 2$ and 3. The metric perturbations transform

$$\begin{aligned} \widehat{h}_{\alpha\beta} &= h_{\alpha\beta} + c_{\alpha,\beta} + c_{\beta,\alpha} + 2\frac{a'}{a^2}b\eta_{\alpha\beta} \\ \widehat{h}_{0\beta} &= \frac{b_{,\beta}}{a} + h_{0\beta}, \qquad \widehat{h}_{00} = 0 \end{aligned} \tag{14}$$

where a prime ($'$) denotes derivative with respect to the time coordinate $\eta = x^0$. The velocity perturbation is gauge invariant to the required order. The density perturbations transform as follows (Brauer 1990),

$$\delta\hat{\rho}_i = \delta\rho_i + \frac{b}{a}\rho_i'. \tag{15}$$

2.1. Energy conservation

The energy conservation (5) for the first-order perturbations of the *dust* reads

$$\left(\frac{\delta\rho_m}{\rho_m}+\frac{1}{2}h\right)'=0 \tag{16}$$

where $h = h^\alpha_\alpha$ is the trace of the metric perturbation and a prime denotes partial derivative with respect to the conformal time η. Hence we get

$$\delta\rho_m = \rho_m \ \left(E(x^\beta) - \tfrac{1}{2}h\right). \tag{17}$$

Here the integration function $E(x^\beta)$ depends only on the space coordinates x^α ($\alpha = 1, 2$ or 3). A gauge transformation (13) alters E as follows,

$$\hat{E}(x^\beta) = \ E(x^\beta) + c^\alpha_{,\alpha}. \tag{18}$$

The energy conservation law (5) for the *radiation* has the form

$$\left(\ \rho_r^{\frac{3}{4}}\sqrt{g}\,u^a_r\ \right)_{,a} = 0. \tag{19}$$

Collecting the first-order terms, we have

$$\left(\frac{3}{4}\frac{\delta\rho_r}{\rho_r}+\frac{1}{2}h\right)' + a\,(\delta u^\alpha_r)_{,\alpha} = 0. \tag{20}$$

2.2. Momentum conservation

For the *dust*, the conservation law (6) simplifies,

$$(ah_{0\alpha})' = 0. \tag{21}$$

This has the solution

$$ah_{\alpha 0} = F_\alpha(x^\beta)\ . \tag{22}$$

The coordinate freedom (13) makes it possible to arrange (White 1973) $F^\alpha_{,\alpha} = 0$, whence

$$h^{0\alpha}{}_{,\alpha} = 0\ . \tag{23}$$

The remaining gauge transformations are still of the form (13), with

$$\Delta b = 0. \tag{24}$$

Here the Laplacian is defined by $\Delta b = -\eta^{\alpha\beta} b_{,\alpha\beta}$.

The momentum conservation law (6) reads for the *radiation* perturbations

$$\left({\rho_r}^{1/4}\delta {u_r}_\alpha\right)' = \tfrac{1}{4}a\rho_r^{-3/4}\delta\rho_{r,\alpha}\ . \tag{25}$$

2.3. The potential v

In (Perjés and Komárik 1995), the potential v has been introduced under the assumption that the vorticity of the radiation vanishes. We now show that this potential exists for generic perturbations, without resorting to any assumption about the vorticity. We introduce the potential $v = v\left(\eta, x^{\beta}\right)$ by writing

$$\tfrac{1}{4} a \rho_r^{-3/4} \delta\rho_r = v'. \tag{26}$$

Under gauge transformations,

$$\hat{v} = v + \rho_r^{1/4} b.$$

Thus we can integrate Eq. (25) as follows:

$$\rho_r{}^{1/4} \delta u_{r\,\alpha} = v_{,\alpha} \tag{27}$$

such that the function of integration has no significance in (26) and is dropped. Hence the velocity perturbation $\delta u_{r\,\alpha} = \delta\left(g_{\alpha b} u_r^b\right)$ has the form

$$\eta_{\alpha\beta} \delta u_r^{\beta} = \left[\rho_r^{-1/4} v_{,\alpha}\left(\eta, x\right) - F_{\alpha}\left(x\right)\right] a^{-2}, \tag{28}$$

Substituting δu_r^{α} and $\delta\rho_r$ in the perturbed energy conservation [Eq.(20)], we obtain

$$3v'' - \Delta v + \tfrac{1}{2} \rho_{ro}^{1/4} a_o h' = 0. \tag{29}$$

3. Einstein equations

The perturbed Einstein tensor will be written

$$G^a{}_b = {}_oG^a{}_b + \delta G^a{}_b\ . \tag{30}$$

Here ${}_oG^a{}_b$ is the unperturbed tensor and $\delta G^a{}_b$ the first-order part. The Einstein equations for the first-order quantities are

$$\delta G^0{}_0 = -\left(\delta\rho_r + \delta\rho_m\right) \tag{31}$$

$$\delta G^{\alpha}{}_0 = -\tfrac{4}{3} a \rho_r \delta u_r^{\alpha} \tag{32}$$

$$\delta G^{\alpha}{}_{\beta} = \tfrac{1}{3} \delta^{\alpha}{}_{\beta} \delta\rho_r. \tag{33}$$

Substitution of the Sachs and Wolfe 1967 expressions for $\delta G^a{}_b$ and separating the trace-free part of the metric perturbation

$$S_{\alpha\beta} = h_{\alpha\beta} - \tfrac{1}{3} \eta_{\alpha\beta} h \tag{34}$$

yields

$$S^{\mu\nu}{}_{,\mu\nu} + \tfrac{2}{3} \Delta h - 2\tfrac{a'}{a} h' \;=\; -2a^2\left(\delta\rho_r + \delta\rho_m\right), \tag{35}$$

$$S^{\alpha\mu}{}_{,\mu}{}' - \tfrac{2}{3} h^{,\alpha\prime} + \Delta h^{0\alpha} - 4\left(2\,\tfrac{a'^2}{a^2} - \tfrac{a''}{a}\right) h^{0\alpha} \;=\tfrac{8}{3} a^3 \rho_r \delta u_r^{\alpha}\ , \tag{36}$$

$$2h'' + 4\tfrac{a'}{a} h' - S^{\mu\nu}{}_{,\mu\nu} - \tfrac{2}{3} \Delta h \;=\; -2a^2 \delta\rho_r\ , \tag{37}$$

$$\begin{aligned}
S^{\alpha}_{\beta}{}'' + 2\frac{a'}{a}S^{\alpha}_{\beta}{}' - \Delta S^{\alpha}_{\beta} \;\; &= \;\; S^{\alpha\mu}{}_{,\beta\mu} + S_{\beta\mu,}{}^{\alpha\mu} - \tfrac{2}{3}\delta^{\alpha}_{\beta}S^{\mu\nu}{}_{,\mu\nu} \\
&\quad + h^{\alpha 0}{}_{,\beta}{}' + h_{\beta 0,}{}^{\alpha}{}' + 2\frac{a'}{a}\left(h^{\alpha 0}{}_{,\beta} + h_{\beta 0,}{}^{\alpha} \right) \\
&\quad - \tfrac{1}{3}h,{}^{\alpha}_{\beta} - \tfrac{1}{9}\delta^{\alpha}_{\beta}\Delta h. \qquad (38)
\end{aligned}$$

The sum of (35) and (37) gives the simple relation

$$h'' + \frac{a'}{a}h' = -a^2\left(2\delta\rho_r + \delta\rho_m\right) \;. \qquad (39)$$

Taking the divergence of Eq. (36) and subtracting the η derivative of (35), we get the integrability condition of these equations. With the help of Eqs. (17), (39) and (20), we may verify, however, that this integrability condition is satisfied identically.

4. Plan of solution

In this section we review the steps of the solution procedure assuming no *a priori* knowledge of the scaling function a. First, we write the radiation velocity perturbation $\delta u_{r\alpha} = \delta\left(g_{\alpha b}u^{b}_{r}\right)$ as

$$\delta u_{r\alpha} = \; a^2\eta_{\alpha\beta}\delta u^{\beta}_{r} + a h_{0\alpha}. \qquad (40)$$

Eliminating δu^{α}_{r} from Eqs (27) and (28), we obtain

$$\left(\frac{3}{2}\frac{\delta\rho_r}{\rho_r} + h\right)'' - \frac{1}{2}\frac{\Delta\delta\rho_r}{\rho_r} = 0. \qquad (41)$$

In the first place, the coupled system consisting of this and Eq. (39) is solved for the radiation density perturbation $\delta\rho_r$ and for the trace h. We next express δu^{β}_{r} from Eqs. (25) and (40):

$$\eta_{\alpha\beta}\delta u^{\beta}_{r} = \frac{1}{4}a^{-2}\rho_r{}^{-1/4}\int a\rho_r^{-3/4}\delta\rho_{r,\alpha}d\eta \; - a^{-1}h_{0\alpha}. \qquad (42)$$

We determine the divergences of the trace-free part from Eqs. (37) and (36) as follows,

$$\begin{aligned}
S^{\mu\nu}{}_{,\mu\nu} \;\; &= \;\; 2h'' + 4\frac{a'}{a}h' \; - \tfrac{2}{3}\Delta h + 2a^2\delta\rho_r \qquad (43) \\
S^{\alpha\mu}{}_{,\mu} \;\; &= \;\; \tfrac{2}{3}h^{,\alpha} - 4\,\frac{a'}{a}\,h^{0\alpha} - \int\left(\Delta h^{0\alpha} + \tfrac{8}{3}a^2\rho_r h^{0\alpha}\right)d\eta \\
&\quad + \tfrac{2}{3}\int a\rho_r{}^{3/4}\int a\rho_r^{-3/4}\delta\rho_r{}^{;\alpha}d\eta_1\; d\eta_2 + S^{\alpha\mu}_{0\;,\mu} \;\; . \qquad (44)
\end{aligned}$$

Here $h^{0\alpha}$ is given by Eq. (22) such that $h^{0\alpha\prime} = -\left(a'/a\right)h^{0\alpha}$. The integration function $S^{\alpha\mu}_{0\;,\mu} = S^{\alpha\mu}_{0\;,\mu}\left(x^{\beta}\right)$ can be further determined by substituting $S^{\alpha\mu}{}_{,\mu}$ in Eq. (43). Upon inserting these expressions in Eq. (38) we get the following inhomogeneous equation for S^{α}_{β}:

$$\begin{aligned}
S^{\alpha}_{\beta}{}'' + 2\frac{a'}{a}S^{\alpha}_{\beta}{}' - \Delta S^{\alpha}_{\beta} \;\; &= \;\; S^{\alpha\mu}_{0\;,\beta\mu} + S_{0\beta\mu,}{}^{\alpha\mu} + h^{;\alpha}_{\;\beta} + \tfrac{1}{3}\delta^{\alpha}_{\beta}\Delta h - 3\,\frac{a'}{a}\left(h^{\alpha 0}{}_{,\beta} + h_{\beta 0,}{}^{\alpha} \right) \\
&\quad + \tfrac{4}{3}\int a\rho_r{}^{3/4}\int a\rho_r^{-3/4}\delta\rho_r{}^{;\alpha}_{\;\beta}d\eta_1\; d\eta_2 \\
&\quad - \int\left[\Delta\left(h^{\alpha 0}{}_{,\beta} + h_{\beta 0,}{}^{\alpha} \right) + \tfrac{8}{3}a^2\rho_r\left(h^{\alpha 0}{}_{,\beta} + h_{\beta 0,}{}^{\alpha} \right)\right]d\eta \\
&\quad - \tfrac{4}{3}\delta^{\alpha}_{\beta}\left(h'' + 2\frac{a'}{a}h' + a^2\delta\rho_r\right). \qquad (45)
\end{aligned}$$

5. k=0 universes

The unperturbed ($h_{ab}=0$) metric satisfies (Misner 1973)

$$3\frac{1}{a^2}\left(\frac{da}{d\eta}\right)^2-\frac{\rho_{m_o}{a_o}^3}{a}-\frac{\rho_{r_o}{a_o}^4}{a^2}=0 \tag{46}$$

such that the density ρ_i equals ρ_{m_o} or ρ_{m_r} at some prescribed conformal time $\eta=\eta_0$. Using the Hubble constant

$$H=\frac{24}{\rho_{m_o}a_0^3}$$

(which is realistic in a matter-dominated ambience at $\eta=\eta_0$), this equation can be written

$$\left(\frac{da}{d\eta}\right)^2-\frac{8}{H}\left(a+\frac{2}{H}\mu^2\right)=0$$

where

$$\mu^2=\frac{\rho_{r_o}a_o}{2\rho_{m_o}}H$$

is a constant.

Hence, fixing the origin of η, the solution has the form

$$a=\tfrac{2}{H}\left(\eta^2-\mu^2\right). \tag{47}$$

Normalizing the time coordinate by

$$\xi=\frac{\eta}{\mu},$$

we have

$$a=\tfrac{2}{H}\mu^2\left(\xi^2-1\right) \tag{48}$$

so that the Big Bang occurs at $\xi=1$. The range of the time coordinate is determined by the ratio of the radiation density to matter density[1], thus in our Universe, $\xi>1$.

The evolution of the trace h, Eq. (39), is driven by the velocity potential [Eq. (26)]:

$$h''+\frac{a'}{a}h'+\frac{12}{Ha}\left(2E(x^\beta)-h\right)=-8a{\rho_r}^{3/4}v' \tag{49}$$

where Eq. (17) was used for the matter density perturbation. We may get rid of the inhomogeneous term $E(x^\beta)$ by introducing the function

$$f=h-2E(x^\beta). \tag{50}$$

Gauge transformations alter f as follows,

$$\hat{f}=f+6\tfrac{a'}{a^2}b. \tag{51}$$

With the new time variable ξ, Eq. (49) takes the form

$$\ddot{f}+\tfrac{2\xi}{\xi^2-1}\dot{f}-\tfrac{6}{\xi^2-1}f=-\tfrac{16}{(\xi^2-1)^2}K\dot{v} \tag{52}$$

---

[1]Small values, $|\xi|<1$, are compatible with a big crunch.

where an overdot means $d/d\xi$ and

$$K = \sqrt{3}\rho_{m_o}\rho_{r_o}^{-3/4}.$$

Equation (29) becomes

$$\ddot{v} - \tfrac{\mu^2}{3}\Delta v + \tfrac{1}{K}\dot{f} = 0. \tag{53}$$

We now introduce a new, *gauge-invariant* potential u by writing

$$u = K\dot{v} + f. \tag{54}$$

Equation (52) and the time derivative of Eq. (53) then are, respectively,

$$\ddot{f} + \tfrac{2\xi}{\xi^2-1}\dot{f} - \left(\tfrac{6}{\xi^2-1} + \tfrac{16}{(\xi^2-1)^2}\right) f = -\tfrac{16}{(\xi^2-1)^2} u \tag{55a}$$

$$\ddot{u} - \tfrac{\mu^2}{3}\Delta\left(u - f\right) = 0. \tag{55b}$$

From (55b), we can express Δf and substitute it in the equation obtained by acting with the Laplacian on Eq. (55a). Thus we get the fourth-order equation for u :

$$\left(\tfrac{d^2}{d\xi^2} + \tfrac{2\xi}{\xi^2-1}\tfrac{d}{d\xi} - \tfrac{6}{\xi^2-1}\right)\Delta u - \tfrac{3}{\mu^2}\left(\tfrac{d^2}{d\xi^2} + \tfrac{2\xi}{\xi^2-1}\tfrac{d}{d\xi} - \tfrac{6}{\xi^2-1} - \tfrac{16}{(\xi^2-1)^2}\right)\tfrac{d^2 u}{d\xi^2} = 0. \tag{56}$$

Given a solution of the coupled equations (55a) and (55b), the radiation density perturbation can be computed from Eq. (26) as follows:

$$\delta\rho_r = \tfrac{4}{\mu a}\rho_r^{3/4}\dot{v} = \tfrac{4}{\mu K}\rho_r^{3/4}\tfrac{u-f}{a}. \tag{57}$$

6. Particular solutions

(i) First we seek solutions with a vanishing gauge-invariant potential, $u = 0$. By Eq. (55b), then f is a harmonic function, $\Delta f = 0$, and Eq. (55a) becomes

$$\ddot{f} + \frac{2\xi}{\xi^2-1}\dot{f} - \left(\frac{6}{\xi^2-1} + \frac{16}{\left(\xi^2-1\right)^2}\right) f = 0. \tag{58}$$

This is the generalized Legendre equation for $\nu = 2$ and $n = 4$. Particular solutions are

$$P_2^4(\xi) = Q_2^4(\xi)\int^{\xi}\frac{1}{(\zeta^2-1)\left[Q_2^4(\zeta)\right]^2}d\zeta$$

and the associated Legendre function of the second kind

$$Q_2^4(\xi) = \left(1-\xi^2\right)^2\frac{d^4 Q_2}{d\xi^4}. \tag{59}$$

Inserting here the Legendre functions $P_2 = \frac{1}{2}\left(3\xi^2 - 1\right)$ and $Q_2 = \frac{1}{2}P_2\ln\frac{\xi+1}{\xi-1} - \frac{3}{2}\xi$, we get the particular solutions in the form

$$f_1 \equiv 48P_2^4 = \frac{1}{5}\frac{\xi^6 - 5\xi^4 + 15\xi^2 + 5}{\left(\xi^2-1\right)^2} \tag{60}$$

$$f_2 \equiv \tfrac{1}{48}Q_2^4 = \frac{\xi}{\left(\xi^2-1\right)^2}.$$

Thus the solution of Eq. (58) is

$$f = A_1\left(x^\alpha\right) f_1 + A_2\left(x^\alpha\right) f_2 \tag{61}$$

where the combination functions A_1 and A_2 are harmonic, $\Delta A_1 = \Delta A_2 = 0$, to yield the property $\Delta f = 0$ as required. We may then perform a gauge transformation (51) with a harmonic b such that $A_2 = 0$ is set in the solution (61).

(*ii*) Our second set of particular solutions arises from the assumption that $\Delta u = 0$. In this case we see from Eq. (56) that it is $\ddot{u}$, rather than f, that satisfies the generalized Legendre equation (58). Hence

$$u = A_1\left(x^\alpha\right) u_1 + A_2\left(x^\alpha\right) u_2 + C\left(x^\alpha\right)\xi + D\left(x^\alpha\right) \tag{62}$$

where the two independent solutions are

$$\begin{aligned} u_1 &\equiv 48 \iint P_2^4 d\xi_1 d\xi_2 = \tfrac{1}{60}\xi^4 - \tfrac{3}{10}\xi^2 - \tfrac{4}{5}\ln(\xi^2-1) \\ u_2 &\equiv \tfrac{1}{48}\iint Q_2^4 d\xi_1 d\xi_2 = \tfrac{1}{4}\ln\tfrac{\xi+1}{\xi-1}. \end{aligned}$$

It follows from $\Delta u = 0$ that each of the integration functions $A_1\left(x^\alpha\right)$, $A_2\left(x^\alpha\right)$, $C\left(x^\alpha\right)$ and $D\left(x^\alpha\right)$ is harmonic.

A particular solution of Eq. (55a) is given by (Bronstein 1974)

$$f_0(\xi) = 16\int \frac{u\left(\zeta\right)}{\zeta^2-1}\left[f_2(\xi)f_1(\zeta) - f_1(\xi)f_2(\zeta)\right]d\zeta.$$

Since u is a harmonic function, so is f_0. Using (60), we obtain

$$\begin{aligned} f_0(\xi) = \tfrac{1}{5}\tfrac{1}{(\xi^2-1)^2}\Big\{&\left[-\tfrac{4}{15}\left(\xi^2-3\right)\xi A_1 + \tfrac{1}{8}\left(3\xi^4-6\xi^2-25\right)A_2 + \left(\xi^4-2\xi^2+5\right)C\right] \\ &\times\left[\left(\xi^2+1\right)\ln\tfrac{\xi-1}{\xi+1} - 2\xi\ln\left(\xi^2-1\right)\right] \\ &+\left[-\tfrac{2}{15}\left(\xi^4-2\xi^2+125\right)A_1 + \tfrac{1}{4}\left(3\xi^2-9\right)\xi A_2 + \left(2\xi^2-6\right)\xi C\right] \\ &\times\left[\left(\xi^2+1\right)\ln\left(\xi^2-1\right) - 2\xi\ln\tfrac{\xi-1}{\xi+1}\right] \\ &-\left(3-\tfrac{213}{5}\xi^2-\tfrac{4}{75}\xi^6+\tfrac{176}{45}\xi^4\right)A_1 + \tfrac{1}{4}\left(7-14\xi^2+3\xi^4\right)\xi A_2 \\ &+2(2\xi^2+\xi^4+5)\xi C + 20\left(1+\xi^2\right)D\Big\}. \end{aligned} \tag{63}$$

The solution of the inhomogeneous Eq. (55a) has the form

$$f = f_0 + F_1\left(x^\alpha\right) f_1 + F_2\left(x^\alpha\right) f_2. \tag{64}$$

Equation (55b) has yet to be satisfied:

$$\ddot{u} + \tfrac{\mu^2}{3}\Delta f = 0.$$

Inserting here (62), we get

$$A_1\left(x^\alpha\right) f_1 + A_2\left(x^\alpha\right) f_2 + \tfrac{\mu^2}{3}\Delta f = 0.$$

Hence

$$A_1 = \tfrac{\mu^2}{3}\Delta F_1, \qquad A_2 = \tfrac{\mu^2}{3}\Delta F_2$$

where F_1 and F_2 are biharmonic functions, *i.e.*, $\Delta\Delta F_1 = \Delta\Delta F_2 = 0$.

(*iii*) We may continue the process of generating new solutions by replacing next the harmonic condition on u with a biharmonic condition. Thus the function $\Delta\Delta f$ satisfies the homogeneous equation (58). The solution of this will define Δu by the equation

$$\Delta\ddot{u} = -\tfrac{\mu^2}{3}\Delta\Delta f. \tag{65}$$

We take the Laplacian of both sides of Eq. (55a) and solve for Δf, given the source term Δu. This in turn yields the potential u by using Eq. (55b). A comparison of (61) and (64) reveals how the solution generating procedure essentially proceeds: the next solution in the sequence is generated by relaxing the harmonic condition on the coefficients of the given solution f.

The general solution of Eq. (49) is

$$h = f_0(\xi) + A_1 f_1(\xi) + A_2 f_2(\xi) + 2E(x^\beta). \tag{66}$$

where $A_1 = A_1(x^\beta)$ and $A_2 = A_2(x^\beta)$ are integration functions. The gauge transformations (14) with a nonvanishing c^α parameter alter h as follows: $\hat{h} = h + 2c^\alpha_{,\alpha}$. Thus, by a suitable gauge transformation, we arrange that $E = 0$, and still we can perform transformations with $c^\alpha_{,\alpha} = 0$.

Finally, we get the trace-free part S^α_β of the metric perturbation from Eq. (38). The solution will have the form

$$S^\alpha_\beta = \underset{0}{S}{}^\alpha_{\ \beta} + \underset{1}{S}{}^\alpha_{\ \beta}$$

where $\underset{0}{S}{}^\alpha_{\ \beta}$ is a *spheroidal wave function* (Flammer 1957, Stratton 1935, Fisher 1937), a solution of the homogeneous equation

$$\ddot{S}^\alpha_\beta + \tfrac{4\xi}{\xi^2-1}\dot{S}^\alpha_\beta - \mu^2\Delta S^\alpha_\beta = 0 \tag{67}$$

and $\underset{1}{S}{}^\alpha_{\ \beta}$ is a particular solution of (38).

Let us consider the solutions which are given by C^∞ functions. Following White 1973, we may then represent the amplitude A_1 in terms of a C^∞ function B as follows,

$$A_1 = \Delta B. \tag{68}$$

The treatment of the curl terms containing $h^{0\alpha}$ is a fairly straightforward task. There remain to be found the pure density perturbations with $h^{0\alpha} = \underset{0}{S}{}^\alpha_{\ \beta} = 0$. For these perturbations, Eq. (38) simplifies somewhat,

$$\begin{aligned}\ddot{S}^\alpha_\beta + \tfrac{4\xi}{\xi^2-1}\dot{S}^\alpha_\beta - \mu^2\Delta S^\alpha_\beta \quad = \quad & \mu^2\left(S^{\alpha\mu}_{0\ ,\beta\mu} + S_{0\beta\mu,}{}^{\alpha\mu} + h^{,\alpha}_{\ \beta} + \tfrac{1}{3}\delta^\alpha_\beta\Delta h\right) \\ & -\tfrac{4}{3}\delta^\alpha_\beta\left(\ddot{h} + \tfrac{4\xi}{\xi^2-1}\dot{h} + \mu^2 a^2\delta\rho_r\right) \\ & +\tfrac{4}{3}\mu^4\int a\rho_r{}^{3/4}\int a\rho_r^{-3/4}\delta\rho_r{}^{,\alpha}_{\ \beta} d\xi_1\ d\xi_2.\end{aligned} \tag{69}$$

For solution (*i*), we have $h = \Delta B f_1$ and $\delta\rho_r = -\frac{4}{\mu K a}\rho_r^{3/4}h$, whence

$$\begin{aligned}\ddot{S}^\alpha_\beta + \tfrac{4\xi}{\xi^2-1}\dot{S}^\alpha_\beta - \mu^2\Delta S^\alpha_\beta \quad = \quad & \mu^2\left(S^{\alpha\mu}_{0\ ,\beta\mu} + S_{0\beta\mu,}{}^{\alpha\mu} + h^{,\alpha}_{\ \beta} - \tfrac{32}{3}\Delta B^{,\alpha}_{\ \beta}\int\tfrac{1}{(\xi^2-1)^2}\int f_1 d\xi_1\ d\xi_2\right) \\ & -\tfrac{8}{3}\delta^\alpha_\beta\Delta B.\end{aligned} \tag{70}$$

We seek a particular solution in the form

$$\underset{1}{S}{}^\alpha_{\ \beta} = \Delta B^{,\alpha}_{\ \beta}F(\xi) - \left(B^{,\alpha}_{\ \beta} + \tfrac{1}{3}\delta^\alpha_\beta\Delta B\right)X(\xi) + S^\alpha_{0\beta} \tag{71}$$

where the requirement of compatibility with the divergence equation (44) yields

$$\begin{aligned} X(\xi) &= f_1 - 8\int \tfrac{1}{(\xi^2-1)^2}\int f_1 d\xi_1\ d\xi_2 \\ &= \tfrac{1}{15}\left[\tfrac{3\xi^4-12\xi^2+1}{\xi^2-1} - 4\ln(\xi^2-1)\right] \end{aligned} \tag{72}$$

and $F(\xi)$ is a function to be determined. By substituting (71) into (70), we get the ordinary differential equation

$$\ddot{F}^{\alpha}_{\beta} + \tfrac{4\xi}{\xi^2-1}\dot{F}^{\alpha}_{\beta} = -\tfrac{8}{3}\mu^2\int \tfrac{1}{(\xi^2-1)^2}\int f_1 d\xi_1\ d\xi_2. \tag{73}$$

This has the solution

$$\begin{aligned} F &= -\tfrac{2}{10125}\tfrac{\mu^2}{\xi^2-1}\left[-63\xi^4 - 1249\xi^2 + \left(45\xi^4+544\xi^2-709\right)\ln(\xi^2-1)\right. \\ &\quad \left. +120\xi\ln\tfrac{\xi-1}{\xi+1} - 120(\xi^2-1)\ln(\xi-1)\ln(\xi+1)\right] \\ &\quad +c_1 + c_2\left(2\tfrac{\xi}{\xi^2-1} + \ln\tfrac{\xi-1}{\xi+1}\right) \end{aligned} \tag{74}$$

where c_1 and c_2 are integration constants. The terms in $\underset{1}{S}{}^{\alpha}_{\ \beta}$ proportional to c_2 have the harmonic amplitude $\Delta B^{;\alpha}_{\ \beta}$. These terms solve the homogeneous equation. They are a special case of gravitational waves described by spheroidal wave functions. This reflects on the ambiguous nature of the decomposition of the perturbations into wave and non-wave parts.

Collecting the results, the tensor perturbation of solution (i) has the form

$$h_{\alpha\beta} = \underset{1}{S}{}_{\alpha\beta} + \tfrac{1}{3}\eta_{\alpha\beta}\Delta B f_1 \tag{75}$$

where $\underset{1}{S}{}_{\alpha\beta}$ and f_1 are given in Eqs. (71) and (60), respectively.

7. The Sachs-Wolfe effect

The temperature variation δT of the cosmic background radiation can be computed (Sachs and Wolfe 1967) as follows,

$$\frac{\delta T}{T} = \frac{1}{2}\int_0^{\eta_R-\eta_E}\left(\frac{\partial h_{\alpha\beta}}{\partial\eta}e^{\alpha}e^{\beta} - 2\frac{\partial h_{0\beta}}{\partial\eta}e^{\beta}\right)dw \tag{76}$$

where η_R and η_E denote the time of reception and emission, respectively, and w is the affine length along the null geodesic of propagation with tangent

$$\frac{dx^a}{dw} = (-1, e^{\alpha}) \tag{77}$$

such that $e^{\alpha}e_{\alpha} = -1$. We consider the contribution of the relatively increasing mode. Then $h_{0\beta} = 0$ and the second term under the integral in Eq. (76) vanishes. The term $\underset{1}{S}{}_{\alpha\beta}$ has the amplitude $B_{,\alpha\beta}$. By using the relation

$$y_{,a}\frac{dx^a}{dw}dw = y_{,\alpha}e^{\alpha}dw - y'd\eta, \tag{78}$$

we get dipole anisotropy contributions with respective amplitudes $B_{,\beta}e^{\beta}$ and $\Delta B_{,\beta}e^{\beta}$ and gravitational redshift terms. The trace part of $h_{\alpha\beta}$, unlike that of the pure dust, is time-dependent, and thus the cancellation of the integrated terms in the (Sachs and Wolfe 1967) result does not occur here. Taken together and excluding dipole anisotropies, the contributions to the temperature variation sum up to

$$\frac{\delta T}{T} = \tau_R - \tau_E + \Delta\tau \tag{79}$$

where τ_R and τ_E are the values of the function

$$\begin{aligned} \tau \quad = \quad & -\tfrac{2}{675}\tfrac{\Delta B}{(\xi+1)^3}(3\xi^2+9\xi^2+13\xi+15)\ln(\xi-1) \\ & -\tfrac{2}{675}\tfrac{\Delta B}{(\xi-1)^3}(3\xi^2-9\xi^2+13\xi-15)\ln(\xi+1) \\ & -\tfrac{1}{15}\tfrac{B}{(\xi^2-1)^3}(3\xi^6-5\xi^4-15\xi^2-15) \\ & +8\tfrac{\Delta B}{(\xi^2-1)^3}\left[c_2\xi-\tfrac{1}{10125}(18\xi^6-220\xi^4-885\xi^2-225)\right] \end{aligned} \tag{80}$$

at the respective events R of reception and E of emission. In addition to these contributions representing the original Sachs-Wolfe effect, we have the integrated term

$$\Delta\tau = \int_0^{\eta_R-\eta_E} \left[\Delta B\tfrac{\partial}{\partial\eta}\left(\tfrac{\partial^2}{\partial\eta^2}F - \tfrac{8}{3}\tfrac{1}{(\xi^2-1)^2}\int f_1 d\xi\right) - B\tfrac{\partial^3}{\partial\eta^3}X\right]dw. \tag{81}$$

At large values of $\eta = \mu\xi$, the function τ increases logarithmically. The integrated term $\Delta\tau$ gives a negligible contribution at late times, but it must be taken into account in the vicinity of $\xi = 1$. This indicates that the temperature fluctuations in solution (i) can be larger than for a pure dust cosmology.

8. Discussion of the results

The evolution of density contrast of the incoherent matter, $\delta\rho_m/\rho_m$ mirrors the evolution of the trace h, as can be seen from Eq. (17). For late times, that is in a matter-dominated era, one should reasonably expect this density contrast to be well-approximated by the Sachs-Wolfe scenario. However, for solution (i), the trace perturbation with the coefficient A_1 behaves asymptotically as $f_1 \propto \xi^2$, unlike the relatively growing mode of Sachs and Wolfe 1967. The perturbation with the coefficient A_2 is $h_2 \propto 1/\xi^3$, precisely as the relatively decreasing mode of Sachs and Wolfe 1967. The trace-free part of solution (i) at late times diverges also faster than in the absence of radiation: it goes like $h_1 \propto \xi^2 \ln\xi$. The asymptotic behavior of solution (ii) is similar: the coefficients of A_2 and C tend to stationary values, and the coefficient of D is asymptotically $\propto 1/\xi^2$.

One may ask whether or not the perturbative cosmologies obtained here are favored by the exact, nonlinear evolutionary processes. This issue may be addressed by numerical methods, to be discussed elsewhere (Czinner 2001).

Acknowledgement

I thank M. Vasúth for illuminating discussions. This work has been supported by the *OTKA* fund T031724.

REFERENCES

Brauer, U., *Zur Störungstheorie der Friedmann-Modelle*, Diplomarbeit, Universität München, 1990.

Bronstein, I. N. and Semendaev, K. A.: *Mathematics*, Technical Publishing House, Budapest, 1974.

Czinner, V. and Perjés, Z., in preparation

Dunsby, P. K. S., 1991, Class. Quantum Grav. **8**, 1785.

Fisher, E., 1937, Philos. Magazine **24**, 245.

Flammer, C.: *Spheroidal wave functions*, Stanford Univ. Press, 1957.

Kodama, H., Sasaki, M., 1987, Int. J. Mod. Phys., **A2**, 491.

Koranda, S. and Allen, B., 1994, astro-ph/9410049

Magueijo, J. C. R., 1993, Phys. Rev. **D15**, R353.

Mather, J. C. *et al.*, 1992, in *Proc. 13th International Conference on General Relativity and Gravitation*, Eds. R. J. Gleiser, C. N. Kozameh and O. M. Moreschi, IOP, Bristol, 1993.

Misner, C. W., Thorne, K. S., Wheeler, J. A., *Gravitation*, 1973, Freeman and Co. San Francisco

Mukhanov, V. F. Feldman, H. A. and Brandenberger, R. H., 1992, Phys. Reports **215**, 203.

Perjés, Z. and Komárik, A., 1995, Int. J. Theor. Phys. **34**, 2275.

Russ, H. *et al.*, 1993, Phys. Rev. **D15**, 4552.

Sachs, R. K. and Wolfe, A. M., 1967, Ap. J. **147**, 73.

Stratton, J. A., 1935, Proc. Nat. Acad. Sci. **21**, 51.

White, C. P., 1973, J. Math. Phys., **14**, 7.

This preprint was prepared with the AAS LATEX macros v4.0.